International Journal of Theoretical Physics, Vol. 34, No. 11, 1995

Oscillatory Behavior in Cosmological Models

A. A. Coley[1] and R. J. van den Hoogen[1]

Received September 12, 1994

Oscillatory behavior in cosmological models is investigated, motivated, in part, by the apparent periodic distribution of galaxies in deep narrow-cone red-shift surveys. In particular, oscillatory behavior in two cosmological models is studied; a qualitative analysis is performed and approximate solutions are found for a soft inflationary model and for a Friedmann–Robertson–Walker model containing a perfect fluid and a scalar field source. These two models are 'conformally equivalent' to particular models arising from a large class of scalar-tensor theories. It is then argued that such oscillatory behavior is a generic property of scalar-tensor theories of gravity.

1. INTRODUCTION

We investigate cosmological models with an oscillatory behavior. This analysis is motivated, in part, by observations from the deep narrow-cone pencil beam surveys of Broadhurst *et al.* (1990), which suggest that the universe may have an oscillatory nature, and this analysis is also related to work of Morikawa (1990a,b) in which the oscillatory behavior in the Hubble parameter was used to model these observations. The cosmological models proposed by Morikawa are not physical, in the sense that they do not agree with all astronomical observations (Hill *et al.*, 1990). However, we want to stress the fact that oscillatory behavior is a general feature of general relativity with a scalar field in particular, and of scalar-tensor theories in general (in which oscillatory behavior is found in the effective gravitational constant as well as in the Hubble parameter). It is this oscillatory behavior in the cosmological models that may account for the apparent periodicity in the Broadhurst *et al.* observations.

[1] Department of Mathematics, Statistics and Computing Science, Dalhousie University, Halifax, Nova Scotia, Canada B3H 3J5.

In Section 2, it is shown that the oscillating behavior of the scalar field in the soft inflationary model found in previous work (Abolghasem *et al.*, 1993) manifests itself in the Hubble parameter, and we also find an approximate solution exhibiting this oscillatory behavior. It is known that in the standard Friedmann–Robertson–Walker (FRW) model with a single scalar field with potential $V(\phi) = (\frac{1}{2}\lambda)\phi^2$ the scalar field undergoes oscillatory behavior (Belinskii *et al.*, 1985a,b; Belinskii and Khalatnikov, 1987). We further expand this result in Section 3 by analyzing an (FRW) model containing a perfect fluid source and a scalar field. Again an approximate solution is found whereby the Hubble parameter is assumed *a priori* to have an oscillatory nature. In Section 4, we argue (using the conformal equivalence between general relativity with multiple minimally coupled scalar fields and scalar-tensor theories of gravity) that this oscillatory behavior is a general property of cosmological models arising from scalar-tensor theories of gravity. In the final section we make some concluding remarks and briefly comment upon the question of whether the oscillatory nature observed in flat FRW models persists in nonflat models and the related question of chaotic behavior in closed models.

2. SOFT INFLATION

2.1. Qualitative Analysis

In a recent paper (Berkin and Maeda, 1991), a soft inflationary scenario was proposed in which the matter content was described by two coupled scalar fields. The action under investigation was

$$S = \int d^4x \sqrt{-g}\left(\frac{R}{2\kappa^2} - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}(\nabla\psi)^2 - e^{-\beta\kappa\phi}V(\psi)\right) \tag{1}$$

where $\kappa^2 = 8\pi G$, ϕ and ψ (the inflaton) are scalar fields, $V(\psi)$ is the potential, and β is a coupling constant. Variation of the action in a flat FRW universe yields the following set of nonlinear second-order ordinary differential equations (ODEs):

$$\ddot{\phi} + 3H\dot{\phi} - \beta\kappa e^{-\beta\kappa\phi}V(\psi) = 0 \tag{2}$$

$$\ddot{\psi} + 3H\dot{\psi} + e^{-\beta\kappa\phi}\frac{dV(\psi)}{d\psi} = 0 \tag{3}$$

where the Friedmann equation is given by

$$H^2 = \frac{\kappa^2}{3}\left(\frac{1}{2}(\dot{\phi})^2 + \frac{1}{2}(\dot{\psi})^2 + e^{-\beta\kappa\phi}V(\psi)\right) \tag{4}$$

An overdot denotes differentiation with respect to time and $H = \dot{a}/a$ is the Hubble parameter, where a is the scale factor.

The system given by equations (2)–(4) was analyzed qualitatively by Abolghasem *et al.* (1993). By defining new independent and dependent variables

$$\frac{dt}{d\tau} = e^{\beta\kappa\phi/2}, \qquad \Phi = \phi' = \dot{\phi}e^{\beta\kappa\phi/2}, \qquad \Psi = \psi' = \dot{\psi}e^{\beta\kappa\phi/2} \tag{5}$$

where prime denotes differentiation with respect to the new time τ, the new evolution equations for Φ' and Ψ' form the following four-dimensional autonomous system of ODEs:

$$\phi' = \Phi \tag{6}$$

$$\psi' = \Psi \tag{7}$$

$$\Phi' = \frac{\beta\kappa}{2}\Phi^2 - \frac{\sqrt{6}\kappa}{2}\Phi\left(\Phi^2 + \Psi^2 + 2V(\psi)\right)^{1/2} + \beta\kappa V(\psi) \tag{8}$$

$$\Psi' = \frac{\beta\kappa}{2}\Psi\Phi - \frac{\sqrt{6}\kappa}{2}\Psi\left(\Phi^2 + \Psi^2 + 2V(\psi)\right)^{1/2} - \frac{dV(\psi)}{d\psi} \tag{9}$$

where

$$H_\tau^2 \equiv \left(\frac{a'}{a}\right)^2 = \frac{\kappa^2}{6}\left(\Phi^2 + \Psi^2 + 2V(\psi)\right) \tag{10}$$

We see that the finite singular points (defined by $\phi' = \psi' = \Phi' = \Psi' = 0$) are given by

$$\Phi = 0, \qquad \Psi = 0, \qquad V(\psi) = 0, \qquad \frac{dV(\psi)}{d\psi} = 0 \tag{11}$$

If we consider a quadratic potential $V(\psi) = (\frac{1}{2}\lambda)\psi^2$, where λ is constant, it follows from equation (11) that there is a nonisolated line singularity along the ϕ axis $(\phi_0, 0, 0, 0)$. Little progress can be made in analyzing these singular points since they are degenerate and nonlinear (Hirsch and Smale, 1974). However, equations (7)–(9) are independent of the variable ϕ; thus we can consider the full system (6)–(9) as a three-dimensional subsystem, given by equations (7)–(9) [determining the qualitative behavior of the three variables (ψ, Φ, Ψ)] together with the fourth equation (6) for ϕ. Hence, determining the qualitative behavior of the three-dimensional subsystem will permit us to determine the asymptotic behavior of the fourth variable ϕ via equation (6).

Progress in determining the qualitative behavior of the subsystem can be made by converting to cylindrical coordinates (Abolghasem *et al.*, 1993):

$$\Psi = r\cos\theta, \qquad \psi = \frac{r}{\sqrt{\lambda}}\sin\theta, \qquad \Phi = z \tag{12}$$

The equations then become

$$r' = r\cos^2\theta\,\frac{\kappa}{2}\left(\beta z - \sqrt{6}(z^2 + r^2)^{1/2}\right) \tag{13}$$

$$\theta' = \sqrt{\lambda} - \cos\theta\sin\theta\,\frac{\kappa}{2}\left(\beta z - \sqrt{6}(z^2 + r^2)^{1/2}\right) \tag{14}$$

$$z' = z\frac{\kappa}{2}\left(\beta z - \sqrt{6}(z^2 + r^2)^{1/2}\right) + \frac{\beta\kappa}{2}r^2\sin^2\theta \tag{15}$$

We note that if $\beta < \sqrt{6}$, then

$$\beta z - \sqrt{6}(z^2 + r^2)^{1/2} \le 0 \tag{16}$$

and we have that $r' \le 0$ everywhere. It can then be shown (Abolghasem *et al.*, 1993) that the equilibrium point ($r = 0$, $z = 0$) is a stable equilibrium point, that is, as $t \to \infty$, $(r, z) \to (0, 0)$. As both r and z approach zero, the dominant part of equation (14) is the first constant term, and thus θ will monotonically increase as $t \to \infty$. Hence the singular point ($r = 0$, $z = 0$) is a stable focus.

2.2. Asymptotic Solution

In a neighborhood of the stable equilibrium point ($r = 0$, $z = 0$) an approximate solution may be found. For small r, z, equation (14) yields $\theta' = \sqrt{\lambda}$, which may be integrated to yield $\theta = \sqrt{\lambda}\tau$ (after normalization). Assuming $z = \alpha r$ (α constant), equations (13) and (15) are consistent if $\beta^2 < 6$ and

$$\alpha = \frac{\beta}{(6 - \beta^2)^{1/2}} \tag{17}$$

(Note that the assumption $z = \alpha r$ is only good for $z \ge 0$, as α must be positive.) The evolution equation for r becomes

$$r' = -\frac{\kappa(6-\beta^2)^{1/2}}{2} r^2 \cos^2(\sqrt{\lambda}\tau) \tag{18}$$

which is integrated to yield a solution for r:

$$\frac{1}{r} - \frac{1}{r_0} = \frac{\kappa(6-\beta^2)^{1/2}}{4}\left[\tau + \frac{1}{2\sqrt{\lambda}}\sin(2\sqrt{\lambda}\tau)\right] \tag{19}$$

From equations (19) and (10) we then obtain (after renormalizing τ)

$$H_\tau = \frac{4}{6-\beta^2}\left[\tau + \frac{1}{2\sqrt{\lambda}}\sin\left(2\sqrt{\lambda}(\tau-\tau_0)\right)\right]^{-1} \tag{20}$$

where

$$\tau_0 = \frac{4}{r_0\kappa(6-\beta^2)^{1/2}}$$

We note the presence of a trigonometric term in the Hubble parameter. It is precisely this term that leads to the oscillatory behavior of the cosmological models.

3. FRIEDMANN–ROBERTSON–WALKER SPACETIME WITH A SCALAR FIELD

3.1. Qualitative Analysis

Next we will use a qualitative analysis to show that this oscillatory type of behavior is possible for an isotropic and spatially homogeneous universe containing a single classical scalar field and noninteracting matter. We shall adopt the 'Ellis-inverse method' (Ellis and Madsen, 1991) to obtain an ad hoc potential $V(\phi)$ corresponding to the desired behavior of the scale factor a.

The field equations of general relativity with a homogeneous scalar field with potential $V(\phi)$ and a matter field in the form of a noninteracting comoving perfect fluid with equation of state $p = (\gamma - 1)\mu$ are

$$H^2 + \frac{k}{a^2} = \frac{\kappa^2}{3}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi) + \mu\right) \tag{21}$$

$$3\dot{H} + 3H^2 = \kappa^2\left(V(\phi) - \dot{\phi}^2 - \frac{1}{2}(3\gamma-2)\mu\right) \tag{22}$$

where $k = \pm 1, 0$, and $\kappa^2 = 8\pi G$. The separate conservation laws are

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0 \tag{23}$$

and

$$\dot{\mu} + 3\gamma H\mu = 0; \qquad \mu = Ma^{-3\gamma} \tag{24}$$

where $V' = dV/d\phi$ and M is a constant. The weak and dominant energy conditions require $M \geq 0$, $0 \leq \gamma \leq 2$, and $V \geq 0$.

In particular, in the qualitative analysis we shall assume that $k = 0$ and that the potential $V(\phi)$ has the form $V(\phi) = \frac{1}{2}\lambda\phi^2$. By defining the new cylindrical coordinates (r, θ, z)

$$\dot{\phi} = r\cos\theta, \qquad \phi = \frac{r}{\sqrt{\lambda}}\sin\theta, \qquad \mu = \frac{1}{2}z^2 \tag{25}$$

the following autonomous system of ODEs results:

$$\dot{r} = -\frac{\sqrt{6}}{2}\kappa r(r^2 + z^2)^{1/2}\cos^2\theta \tag{26}$$

$$\dot{\theta} = \sqrt{\lambda} + \frac{\sqrt{6}}{2}\kappa(r^2 + z^2)^{1/2}\cos\theta\sin\theta \tag{27}$$

$$\dot{z} = -\frac{\sqrt{6}}{4}\kappa\gamma z(r^2 + z^2)^{1/2} \tag{28}$$

The singular point at finite values of the variables is given by ($r = 0$, $z = 0$). This singular point is easily seen to be a sink, because $\dot{r} \leq 0$, and for $z > 0$, $\dot{z} < 0$ and for $z < 0$, $\dot{z} > 0$. As both r and z approach zero the dominant part of equation (27) is the first constant term, and thus θ will monotonically increase as $t \to \infty$. Hence the singular point ($r = 0$, $z = 0$) is a stable focus for the $k = 0$ FRW models.

3.2. Asymptotic Solution

A simple Painlevé analysis using the ARS algorithm (Ablowitz *et al.*, 1980; Ramani *et al.*, 1989) indicates that the system of equations (22)–(24), [for general k and a potential of the form $V(\phi) = \frac{1}{2}\lambda\phi^2$] does not have the Painlevé property, which is conjectured to be a necessary condition for integrability (Ablowitz *et al.*, 1980; Ramani *et al.*, 1989), and hence an exact solution may not exist. Consequently, we shall seek an approximate solution.

Again, in a neighborhood of the singular point ($r = 0$, $z = 0$) an approximate solution may be found. Using the 'Ellis-inverse method,' and in analogy with (20), we assume that the Hubble parameter is of the following form:

$$H = \alpha\left(\frac{1}{t} - \frac{\beta}{t^2}\sin(bt)\right) \tag{29}$$

where α, β, and b are constants and we choose units so that $8\pi G = \kappa^2 = 1$. From equation (29) we have that

$$a = a_0 t^\alpha\left(1 + \frac{\alpha\beta}{bt^2}\cos(bt) + O\left(\frac{1}{t^3}\right)\right) \tag{30}$$

$$\dot{H} = -\frac{\alpha}{t^2}\left(1 + b\beta\cos(bt)\right) + O\left(\frac{1}{t^3}\right) \tag{31}$$

where $O(1/t^3)$ denotes terms of order $(1/t^3)\cdot$(trigonometric functions). From (24) we have that

$$\mu = mt^{-2} + O\left(\frac{1}{t^4}\right) \tag{32}$$

where $m \equiv Ma_0^{-3\gamma}$ and we have chosen $\alpha \equiv 2/(3\gamma)$.

From equations (21) and (22) we find that

$$\dot{\phi}^2 = 2\frac{k}{a^2} - 2\dot{H} - \gamma m t^{-2} \tag{33}$$

Henceforth we shall assume that $k = 0$, whence

$$\dot{\phi}^2 = \frac{1}{t^2}\left((2\alpha - \gamma m) + 2\alpha b\beta\cos(bt)\right) \tag{34}$$

$$= \frac{\delta^2}{t^2}\left(1 + \cos(bt)\right) \tag{35}$$

where $\delta^2 \equiv 2\alpha - \gamma m = 2\alpha b\beta$ (which serves to define m). Thus

$$\dot{\phi} = \frac{\sqrt{2}\delta}{t}\cos\left(\frac{b}{2}t\right) \tag{36}$$

and hence

$$\phi = \phi_0 + 2\frac{\sqrt{2}\delta}{bt}\sin\left(\frac{b}{2}t\right) + O\left(\frac{1}{t^2}\right) \tag{37}$$

which serves to define $t = t(\phi)$.

Finally, from equations (21), (22), and (37), we obtain the potential $V(\phi)$:

$$V(\phi) = \dot{H} + 3H^2 + \frac{1}{2}(\gamma - 2)\mu \tag{38}$$

$$= \left(3\alpha^2 - \alpha - \alpha b\beta + \frac{1}{2}(\gamma - 2)m\right)\frac{1}{t^2} + \frac{2\alpha b\beta}{t^2}\sin^2\left(\frac{b}{2}t\right) \tag{39}$$

$$= \delta^2\frac{1-\gamma}{\gamma}\frac{1}{t(\phi)^2} + \frac{b^2}{8}(\phi - \phi_0)^2 \tag{40}$$

To leading order, equations (21), (22), and equation (23) are satisfied. In general, we obtain the desired behavior for $H(t)$ [viz. (29)] for a single scalar field with potential (40) which has a quadratic part and an 'additional part'. Note, however, that in the case $\gamma = 1$ the first term in (40) is absent; that is, the potential $V(\phi)$ will be a simple quadratic function of the scalar field and the energy density μ will be of the form $\mu = mt^{-2}$, where $m = \frac{4}{3}(1 - b\beta)$.

4. SCALAR-TENSOR THEORIES

In the previous two sections we demonstrated that in two cosmological models the late-time asymptotic behavior (that is, as $t \to \infty$) of the Hubble parameter contains trigonometric contributions. We will argue that this oscillatory behavior is a rather general property in the class of scalar-tensor theories of gravity.

The action in the so-called Bergmann–Wagoner theories of gravity (Will, 1981) can be written in the form

$$S = \int d^4x\,\sqrt{-g}\left\{\phi R - \frac{w(\phi)}{\phi}(\nabla\phi)^2 - 2\phi\lambda(\phi) + L_m\right\} \tag{41}$$

where L_m is the Lagrangian due to matter and other nongravitational fields. Action (41) is equivalent (up to field redefinitions) to

$$S = \int d^4x\,\sqrt{-g}\left\{f(\Phi)R - \frac{1}{2}(\nabla\Phi)^2 - V(\Phi) + L_m\right\} \tag{42}$$

where

$$\phi = f(\Phi), \qquad w(\phi) = \frac{f(\Phi)}{2f'(\phi)^2}, \qquad \lambda(\phi) = \frac{V(\Phi)}{2f(\Phi)} \tag{43}$$

One of the well-known examples of a scalar-tensor theory of gravity is the subclass where

$$f(\Phi) = \frac{\zeta\Phi^2}{16\pi} \quad \text{and} \quad V(\Phi) \equiv 0$$

or equivalently

$$w(\phi) = \frac{2\pi}{\zeta} \quad \text{and} \quad \lambda(\phi) \equiv 0$$

which results in the standard Brans–Dicke theory of gravity. The benefits of using either the action (41) or the action (42) are discussed in Liddle and Wands (1992).

We note that if $L_m \equiv 0$, then both (41) and (42) can be recast into the form of general relativity minimally coupled to a scalar field through conformal transformations and field redefinitions ((Magnano and Sokolowski, 1993; Walliser, 1992) and references therein). If $L_m = -\frac{1}{2}(\nabla\psi)^2 - V(\psi)$, whence the matter is due to a second scalar field, then (41) and (42) may be recast into the form of a soft inflationary scenario again through a conformal transformation and field redefinitions (Berkin and Maeda, 1991). It was precisely these two cases that were investigated in the previous two sections of this paper. In Sections 2 and 3 we observed that asymptotically the Hubble parameter contained trigonometric contributions. Therefore, in scalar-tensor theories, the conformally related Hubble parameter might also be expected to contain trigonometric contributions in general.

However, these conformal transformations may lead to problems; for example, the metric may change signature (Magnano and Sokolowski, 1993), or the conformal transformation may become singular at the critical points of the field equations (Walliser, 1992). Therefore, general results using qualitative theory are problematic. Consequently, we shall simply demonstrate the genericity of the oscillatory behavior of these models with the above action (41) or (42), by briefly discussing previous work done without utilizing a conformal transformation.

Walliser (1992) studied the field equations resulting from the action

$$S = \int d^4x \sqrt{-g}\left\{\frac{1}{6} h(\phi)R + g(\phi)(\nabla\phi)^2 - V(\phi) + L_m\right\}$$

which is a further generalization of the action (42), in a flat Robertson–Walker background. He found that at finite values, there is a critical point that is a stable focus, and near this critical point the oscillatory behavior in the variables manifests itself in both the Hubble parameter and the effective gravitational constant. Romero and Barros (1993) investigated a class of vacuum Brans–Dicke models. They found that for appropriate values of the parameter w, the late-time asymptotic behavior may be oscillatory in nature, whence the effective gravitational constant will also oscillate. [Note in this case the potential $\lambda(\phi) \equiv 0$]. Scalar-tensor theories of gravity based upon the action (42) with a nonminimal coupling function $f(\Phi)$ of the form

$$f(\Phi) = \frac{1}{16\pi G} - \frac{1}{2}\zeta\Phi^2 \tag{44}$$

have also been investigated recently (Morikawa, 1990a,b; Barroso *et al.*, 1992; Futamase and Maeda, 1989). Barroso *et al.* (1992) have shown that for finite values, the critical points exhibit an oscillatory nature. It is precisely these models that Morikawa (1990a,b) investigated in an attempt to model the periodic distribution of galaxies.

However, oscillatory behavior is found not only in scalar-tensor theories of gravity, but also in more general theories of gravity, such as theories in which derivatives of the scalar fields are nonminimally coupled to the curvature R via an action of the form

$$S = \int d^4x\,\sqrt{-g}\left\{\left[\frac{1}{16\pi G} - \zeta f(\Phi) - \eta(\nabla\Phi)^2\right]R - (\nabla\Phi)^2 + V(\Phi) + L_m\right\}$$

where ζ and η are constants and where $f(\Phi)$ is an arbitrary function of Φ (Amendola, 1993; Amendola *et al.*, 1993), and modified theories of gravity with an action of the form

$$S = \int d^4x\,\sqrt{-g}\{F(R, R_{\mu\nu}R^{\mu\nu}, C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta}, \ldots)\}$$

where F is an arbitrary function of its arguments (Brandenberger *et al.*, 1993). This indicates that oscillatory behavior in alternative theories of gravity may be a generic property. The behavior of such cosmological models in alternative theories of gravity, including more general theories than those discussed above, is further investigated in van den Hoogen (1995).

It is also interesting to note that the deceleration parameter q, defined by

$$q \equiv -\frac{\ddot{a}a}{\dot{a}^2} = -\frac{1}{H^2}(\dot{H} + H^2) \tag{45}$$

changes sign periodically. A negative q indicates that there exists a region of phase space with an accelerated expansion; that is, inflation occurs. In the soft inflation case of Section 2 it was shown in van den Hoogen (1994a) and Abolghasem *et al.* (1993) that for $\beta^2 < 2$ the model must undergo periods of both accelerated and decelerated expansion. For the asymptotic solution (20) it is easy to see that q has the form

$$q = \frac{2 - \beta^2}{4} + \frac{6 - \beta^2}{4} \cos[2\sqrt{\lambda}(\tau - \tau_0)] \tag{46}$$

which has an oscillatory nature. For the asymptotic solution (29) the deceleration parameter is given by

$$q = \frac{1 - \alpha}{\alpha} + \frac{b\beta}{\alpha} \cos(bt) \tag{47}$$

which is again easily seen to have an oscillatory behavior (van den Hoogen, 1995).

The oscillatory behavior in (both the Hubble parameter H and) the deceleration parameter q implies that the universe expands faster in some stages (and slower in others) than its average value (which is the same as in the 'nonoscillating' case) (Steinhardt, 1991). Maeda (1991) studied such oscillatory models with regard to structure formation and found a significant enhancement in the growth of density perturbations, which perhaps further motivates the study of such models. In other work Futamase and Maeda (1989) studied scalar-tensor theories with a nonminimal coupling function of the form (44) and found that there exist severe restrictions on the parameter ζ in order for inflation to occur, and suggested that adding a second minimally coupled scalar field might give rise to a more realistic model.

5. CONCLUSION

It was proposed by Morikawa (1990a,b) that an oscillating Hubble parameter may be responsible for the apparent periodic distribution of galaxies (Broadhurst *et al.*, 1990). However, Hill *et al.* (1990) argued that such oscillations in the Hubble parameter are not consistent with the observations of Broadhurst *et al.* within the standard FRW model of general relativity. In addition they also argued that Morikawa's models are not physically viable.

However, according to Hill *et al.* (1990), an oscillating gravitational constant G is one of the most viable candidates for generating an apparent periodicity in the distribution of galaxies. If the gravitational constant G is allowed to vary with respect to time in the standard FRW models, then oscillations in the Hubble parameter induce oscillations in the gravitational constant G. In this case, the resulting model nearly agrees with the Broadhurst *et al.* results and the shortfalls may be due to errors in determining the quantity $\dot{G}/G$ from the Viking experiment (Hill *et al.*, 1990).

Morikawa (1990a,b), Hill *et al.* (1990), and Steinhardt (1991) suggested that one way in which the gravitational constant G may vary is to introduce a scalar field that is nonminimally coupled to the curvature R in the Lagrangian (in other words, to introduce a scalar-tensor theory of gravity). In Morikawa

(1990a,b) and Hill *et al.* (1990) the nonminimal coupling function has the form (44), and consequently the effective gravitational constant G_{eff} is given by

$$16\pi G_{\text{eff}} = \left(\frac{1}{16\pi G} - \frac{1}{2}\zeta\Phi^2\right)^{-1} \tag{48}$$

and thus varies with time. This class of theories constitutes a subclass of the larger class of scalar-tensor theories of gravity governed by (42) in which the effective gravitational constant varies with time according to

$$16\pi G_{\text{eff}} = f(\Phi)^{-1} \tag{49}$$

[or, equivalently, $16\pi G_{\text{eff}} = \phi^{-1}$ from (41)]. Therefore, these more general scalar-tensor theories of gravity may give rise to physically acceptable cosmological models.

Both of the models that we have analyzed are flat FRW models and the discussion has focused upon such zero-curvature cosmologies. It is of interest to ask whether the oscillatory nature of the $k = 0$ FRW models is stable to perturbations in the curvature; that is, whether this oscillatory behavior persists in the $k \neq 0$ models. Belinskii (1985a,b; Belinskii and Khalatnikov, 1987) studied the FRW models with a minimally coupled scalar field and a potential of the form $V(\phi) = \frac{1}{2}\lambda\phi^2$ using qualitative analysis. They found that for the $k = 0$ and $k = -1$ models the oscillatory behavior is a general feature; however, in the $k = +1$ case they noted the existence of a closed chain of trajectories, which hints at the possibility of the system having periodic orbits. Hawking (1984) showed that there do indeed exist periodic orbits in such models and, in particular, a countable infinite set of periodic orbits without singularity. Further, Page (1984) showed that there also exists a discrete set of nonperiodic orbits without a singular point. These two properties suggest the possibility of chaos in the closed models.

We note that the Painlevé analysis discussed in Section 3 concerning the integrability of the minimally coupled FRW model also suggests chaos. Recently, Calzetta (1994) studied various cosmological models using Melnikov's method. In particular, Calzetta analyzed a class of scalar-tensor theories with $f(\Phi)$ of the form (44) with $\zeta = 1/6$ (the conformally coupled case) and with a potential of the form $V(\Phi) = \frac{1}{2}\lambda\Phi^2$ and found, using both Melnikov's method and numerical techniques, that the $k = +1$ FRW models exhibit chaotic behavior (which in turn suggests the nonintegrability of the models). Clearly it is of interest to study whether closed FRW models with a potential of the form $V(\Phi) = \frac{1}{2}\lambda\Phi^2$ exhibit chaotic behavior in other theories of gravity.

In closing, we have studied two models that exhibit oscillatory behavior. In both the soft inflationary model and in the FRW model with a scalar field, the Hubble parameter was found to contain trigonometric contributions asymptotically. Using the fact that these two theories are conformally equivalent to particular scalar-tensor theories of gravity (up to field redefinitions), we have argued that such oscillatory behavior is a general property of models in all scalar-tensor theories of gravity. We also remarked that this oscillatory behavior is found not only in general relativity and in scalar-tensor theories of gravity, but also in other alternative theories of gravity. With the deep narrow-cone pencil-beam red-shift surveys exhibiting an apparent oscillatory behavior in the observed universe, these oscillatory cosmological models merit further investigation.

ACKNOWLEDGMENT

This research was funded by the Natural Sciences and Engineering Research Council of Canada.

REFERENCES

Ablowitz, M. J., Ramani, A., and Segur, H. (1980). *Journal of Mathematical Physics,* **21**, 715–721.

Abolghasem, G., Burd, A., Coley, A., and van den Hoogen, R. (1993). *Physical Review D,* **48**, 557–561.

Amendola, L. (1993). *Physics Letters B,* **301**, 175–182.

Amendola, L., Bellisai, D., and Occhionero, F. (1993). *Physical Review D,* **47**, 4267–4272.

Barroso, A., Casasayas, J., Crawford, P., Moniz, P., and Nunes, A. (1992). *Physical Letters B,* **275**, 264–272.

Belinskii, V. A., Grishchuk, L. P., Khalatnikov, I. M., and Zel'dovich, Y. B. (1985a). *Physics Letters B,* **155**, 232–236.

Belinskii, V. A., Grishchuk, L. P., Zel'dovich, Y. B., and Khalatnikov, I. M. (1985b). *Soviet Physics JETP,* **62**, 195–203.

Belinskii, V. A., and Khalatnikov, I. M. (1987). *Soviet Physics JETP,* **66**, 441–449.

Berkin, A. L., and Maeda, K. (1991). *Physics Review D,* **44**, 1691–1704.

Brandenberger, R., Mukhanov, V., and Sornborger, A. (1993). *Physical Review D,* **48**, 1629–1642.

Broadhurst, T. J., Ellis, R. S., Koo, D. C., and Szalay, A. S. (1990). *Nature,* **343**, 726–728.

Calzetta, E. (1994). In *Proceedings of the NATO Advanced Research Workshop on Deterministic Chaos in General Relativity,* D. Hobill, A. Burd, and A. Coley, eds., Plenum, New York.

Ellis, G. F. R., and Madsen, M. S. (1991). *Classical and Quantum Gravity,* **8**, 667.

Futamase, T., and Maeda, K. (1989). *Physical Review D,* **39**, 399–404.

Hawking, S. W. (1984). In *Relativity, Groups and Topology II,* B. Dewitt and R. Stora, eds., North-Holland, Amsterdam.

Hill, C. T., Steinhardt, P. J., and S. Turner, M. (1990). *Physics Letters B,* **252**, 343–348.

Hirsch, M. W., and Smale, S. (1974). *Differential Equations, Dynamical Systems, and Linear Algebra,* Academic Press, New York.

Liddle, A., and Wands, D. (1992). *Physical Review D,* **45**, 2665.

Maeda, K. (1991). In *Proceedings of the Sixth Marcel Grossman Meeting on General Relativity*, H. Sato and T. Nakamura, eds., World Scientific, Singapore.

Magnano, G., and Sokolowski, L. M. (1993). Preprint.

Morikawa, M. (1990a). *Astrophysical Journal*, **362**, L37–L39.

Morikawa, M. (1990b). *Astrophysical Journal*, **369**, 20–29.

Page, D. N. (1984). *Classical and Quantum Gravity*, **1**, 417–427.

Ramani, A., Grammaticos, B., and Bountis, T. (1989). *Physics Reports*, **180**, 159–245.

Romero, C., and Barros, A. (1993). *General Relativity and Gravitation*, **25**, 491–502.

Steinhardt, P. J. (1991). In *Proceedings of the Sixth Marcel Grossmann Meeting on General Relativity*, H. Sato and T. Nakamura, eds., World Scientific, Singapore.

van den Hoogen, R. (1994a). In *Proceedings of the Fifth Canadian Conference on General Relativity and Relativistic Astrophysics*, R. B. Mann and R. G. McLenaghan, eds., World Scientific, Singapore.

van den Hoogen, R. (1995). Qualitative analysis of cosmological models, Ph.D. Thesis, Dalhousie University, Halifax.

Walliser, D. (1992). *Nuclear Physics B*, **378**, 150–174.

Will, C. M. (1981). *Theory and Experiment in Gravitational Physics*, Cambridge University Press, Cambridge.